

Noise, risk premium, and bubble

Grzegorz Andruszkiewicz

Dorje C. Brody

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Department of Mathematics, Imperial College London, London SW7 2AZ, UK

Abstract

The existence of the pricing kernel is shown to imply the existence of an ambient information process that generates market filtration. This information process consists of a signal component concerning the value of the random variable X that can be interpreted as the timing of future cash demand, and an independent noise component. The conditional expectation of the signal, in particular, determines the market risk premium vector. An addition to the signal of any term that is independent of X , which generates a drift in the noise, is shown to change the drifts of price processes in the physical measure, without affecting the current asset price levels. Such a drift in the noise term can induce anomalous price dynamics, and can be seen to explain the mechanism of observed phenomena of equity premium and financial bubbles.

1 Introduction

The market risk premium is one of the main factors that drives the return of any given portfolio of assets. Hence it is a key quantity for hedge funds, pension funds, and numerous other investors. The risk premium can make investments grow smoothly or jump up and down widely, often in an unpredictable manner. In spite of its importance in asset allocation, however, the risk premium is notoriously difficult to estimate from observed price processes of various risky assets (see, e.g., Rogers 2001). Is it possible then to estimate the risk premium from current prices of financial derivatives?

If $\{S_t\}$ denotes the price process of a risky asset and $h(s)$ is the payout function of a European contingent claim expiring at T , then the price of this derivative is given by the expectation of the cash flow $h(S_T)$, suitably discounted, in the risk-neutral measure. Because asset price processes in the risk-neutral measure are independent of the market risk premium, one might be tempted to conclude therefore that derivative prices are likewise independent of the risk premium. Indeed, in the case of the Black-Scholes-Merton model where all relevant parameters are constant in time, the risk premium parameter essentially drops out of various derivative pricing formulae. Notwithstanding this example, it is worth bearing in mind that the choice of the pricing measure does depend on the choice of the risk premium. Thus, derivative prices in general will depend implicitly on the risk premium, often in a nonlinear way. It follows that calibration of the market risk premium from option prices is feasible within a given modelling framework (Brody *et al.* 2011).

The main purpose of the present paper is to address the question whether it is possible, at least in principle, to determine the risk premium unambiguously, if the totality of arbitrage-free market prices for various derivatives were available. We shall find that the market risk premium consists of two components in an additive manner (for models based on Brownian filtrations): The first of the two, which we might call a ‘systematic’ component, depends explicitly on the term structure of the market, while the second, which we might call an ‘idiosyncratic’ component, is independent of the term structure of the market, and thus can be identified as *pure noise*. We show that the systematic component can in principle be determined from current market data, whereas the idiosyncratic noise component is strictly ‘hidden’ and thus cannot be inferred from derivative prices. Therefore, the risk premium can be backed out from market data only up to an indeterminable additive noise.

Although the noise component cannot be inferred directly, it nevertheless has an impact on the dynamics of asset prices under the physical measure, even though it does not reflect the ‘true’ state of affairs. Hence a spontaneous creation of superfluous noise can move the price of an asset in an essentially arbitrary direction. In particular, because the risk premium, and hence its noise component, is a vectorial quantity, the direction of the noise vector can at times lie close to the directions of volatility vectors of the share prices of a particular industrial sector, leading to the creation of a ‘bubble’ for that sector by pushing up those share prices. When a more reliable information concerning the state of that sector is unveiled, the direction of the risk premium vector is likely to change so as to generate a negative component in the excess rate of return. This can be exacerbated by an increase in the magnitude of asset volatilities due to information revelation, thus leading to a ‘burst’. Such a scenario need not be confined to a particular financial sector; the existence of the so-called ‘equity premium puzzle’ over a specified period can likewise be attributed to the prevailing noise that points in the general direction of the equity market volatility, but not in the direction of the bond market volatility.

Needless to say, our formulation does not explain the cause of the creation of anomalous price movements such as a financial bubble or an equity premium; nor does it address the predictability of these events. In fact, according to our characterisation, bubbles can at best be identified retrospectively, after their bursts. Nevertheless, we are able to describe the mechanism by which such anomalous price movements are generated in a simple and intuitive manner. In particular, since our characterisation of a bubble is different from those more commonly used in the literature, we are able to circumvent the analysis based on subtle distinctions between local and true martingales. We also provide a heuristic argument why the hidden noise might have the tendency of creating equity premium.

2 Pricing kernel

For definiteness, we shall be adopting the pricing kernel approach (see, e.g., Cochrane 2005, Björk 2009). We model the financial market on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with filtration $\{\mathcal{F}_t\}_{t \geq 0}$. Here \mathbb{P} denotes the ‘physical’ probability measure, and $\{\mathcal{F}_t\}$ is assumed to be generated by a multi-dimensional Brownian motion. Expectation under \mathbb{P} is denoted $\mathbb{E}[-]$, and for the conditional

expectation with respect to \mathcal{F}_t we write $\mathbb{E}_t[-]$. Two other probability measures enter the ensuing discussion; these are the risk-neutral measure \mathbb{Q} and an auxiliary measure \mathbb{R} to be described below. Expectations in these measures will be written $\mathbb{E}^{\mathbb{Q}}[-]$ and $\mathbb{E}^{\mathbb{R}}[-]$, respectively.

We assume that the market is free of arbitrage opportunities, and that there is an established pricing kernel, but market completeness is not assumed. These assumptions imply the existence of a unique preferred risk-neutral measure \mathbb{Q} . The pricing kernel, denoted here by $\{\pi_t\}_{t \geq 0}$, is a positive supermartingale with the property that if S_T is the price at time T of an asset that pays no dividend, then the price at time t of the asset is given by

$$S_t = \frac{1}{\pi_t} \mathbb{E}_t[\pi_T S_T]. \quad (1)$$

In particular, if $S_T = 1$, then (1) gives the pricing formula for the discount bond: $P_{tT} = \mathbb{E}_t[\pi_T]/\pi_t$.

We shall proceed by discussing some properties of the pricing kernel that are relevant to our analysis here. In addition to being a positive supermartingale, the pricing kernel fulfils the condition that $\mathbb{E}[\pi_t] \rightarrow 0$ as $t \rightarrow \infty$. A positive supermartingale possessing this property is known as a *potential*. It follows that every pricing kernel can be represented as a potential, and conversely every potential constitutes an admissible pricing kernel. The Doob-Meyer decomposition then shows that $\{\pi_t\}$ can be represented uniquely in the form:

$$\pi_t = \mathbb{E}_t[A_\infty] - A_t, \quad (2)$$

where $\{A_t\}_{t \geq 0}$ is an increasing adapted process such that A_∞ is finite. Note that $\mathbb{E}_t[A_\infty]$ is a uniformly integrable martingale. We may define $\{A_t\}$ according to

$$A_t = \int_0^t a_s ds \quad (3)$$

for some adapted nonnegative process $\{a_t\}$. Hence it suffices to choose the process $\{a_t\}$ to model the pricing kernel, and this leads to the potential approach of Rogers (1997) to model term structure dynamics. A substitution shows that

$$\pi_t = \int_t^\infty \mathbb{E}_t[a_u] du. \quad (4)$$

The representation (4) resembles that of Flesaker and Hughston (1996,1997), if we make the following identification. First, writing $\rho_0(T) = -\partial_T P_{0T}$, where P_{0T} is the initial discount function, we see that the processes $\{M_t(u)\}_{t \geq 0, u \geq t}$ defined by

$$M_t(u) = \frac{\mathbb{E}_t[a_u]}{\rho_0(u)} \quad (5)$$

is a one-parameter family of positive martingales, i.e. for each fixed $u \geq t$, $\{M_t(u)\}$ is a martingale. This follows on account of the martingale property of the conditional expectation $\mathbb{E}_t[a_u]$. In terms of these positive martingales, the pricing kernel can be expressed in the Flesaker-Hughston form:

$$\pi_t = \int_t^\infty \rho_0(u) M_t(u) du. \quad (6)$$

From the martingale representation theorem we deduce that the dynamical equations satisfied by the positive martingale family $\{M_t(u)\}$ take the form:

$$dM_t(u) = M_t(u)v_t(u)d\xi_t, \quad (7)$$

where $\{v_t(u)\}$ is a family of adapted (in general vectorial) processes and $\{\xi_t\}$ is a standard multi-dimensional Brownian motion under the \mathbb{P} measure. We observe therefore that modelling the pricing kernel is equivalent to modelling the one-parameter family of volatility processes $\{v_t(u)\}$. On account of (6) and (7) we deduce, by an application of Ito's lemma, that

$$\frac{d\pi_t}{\pi_t} = -r_t dt - \lambda_t d\xi_t, \quad (8)$$

where

$$r_t = \frac{\rho_0(t)M_t(t)}{\int_t^\infty \rho_0(u)M_t(u)du} \quad (9)$$

is the short rate, and

$$\lambda_t = -\frac{\int_t^\infty \rho_0(u)v_t(u)M_t(u)du}{\int_t^\infty \rho_0(u)M_t(u)du} \quad (10)$$

is the market risk premium. The fact that the drift of $\{\pi_t\}$ can be identified with the short rate can be seen by applying the martingale condition (1) on the money market account $\{B_t\}$ satisfying $dB_t = r_t B_t dt$. That is, the drift of $\{\pi_t B_t\}$ vanishes if and only if the drift of $\{\pi_t\}$ is $\{-r_t\}$. Similarly, let us write $\{\mu_t\}$ for the drift of a risky asset $\{S_t\}$ that pays no dividend, and $\{-\lambda_t\}$ for the volatility of $\{\pi_t\}$. Then the martingale condition on $\{\pi_t S_t\}$ implies that $\mu_t = r_t + \lambda_t \sigma_t$, which shows that $\{\lambda_t\}$ indeed expresses the excess rate of return above the risk-free rate in unit of volatility.

An advantage of working with the pricing kernel is that once a model is chosen for the volatility processes $\{v_t(u)\}$ of the martingale family, we are able not only to price a wide range of derivatives via the pricing formula $\mathbb{E}[\pi_T H_T]$, where H_T is the payout of a derivative, but also to obtain a model for the interest rate term structure. Furthermore, a model for $\{v_t(u)\}$, which can be calibrated by use of market data for derivative prices, implies a process for the risk premium $\{\lambda_t\}$ according to the prescription (10), and this in turn can be used for asset allocation purposes. This is the sense in which derivative prices can be used to calibrate the risk premium, within any modelling framework (cf. Brody *et al.* 2011). The issues that we would like to address here are: (a) the ambiguity associated with the determination of the risk premium from market data; and (b) the identification of the origin of this ambiguity. For these purposes, it is useful to examine the probabilistic characterisation of the pricing kernel, within the term structure density approach of Brody and Hughston (2001).

3 Probabilistic representation of the pricing kernel

To proceed, we shall make the following observation that the positivity of nominal interest and the requirement that a bond with infinite maturity must have

vanishing value imply that $\rho_0(T) = -\partial_T P_{0T}$ defines a probability density function on the positive half-line (Brody and Hughston 2001). More generally, the positivity of the martingale family $\{M_t(u)\}$ implies that $\{\rho_t(u)\}$ defined by

$$\rho_t(u) = \frac{\rho_0(u)M_t(u)}{\int_0^\infty \rho_0(u)M_t(u)du} \quad (11)$$

is a measure-valued process, i.e. $\rho_t(u) \geq 0$ for all t and all u ; and

$$\int_0^\infty \rho_t(u)du = 1 \quad (12)$$

for all $t \geq 0$. The measure-valued process thus introduced suggests the existence of a random variable X whose conditional density under some probability measure is given by (11). Furthermore, an application of Ito's lemma on (11) shows that

$$\frac{d\rho_t(u)}{\rho_t(u)} = (v_t(u) - \hat{v}_t) (d\xi_t - \hat{v}_t dt), \quad (13)$$

where

$$\hat{v}_t = \int_0^\infty v_t(u)\rho_t(u)du \quad (14)$$

can be thought of as the conditional expectation of $v_t(X)$.

Indeed, the dynamical equation (13) takes the form of a Kushner equation, thus implies the existence of the following auxiliary filtering problem. For simplicity of exposition, let us for now assume that the one-parameter family of volatility processes $\{v_t(u)\}$ is deterministic. We introduce a probability space $(\Omega, \mathcal{F}, \mathbb{R})$, upon which X is a positive random variable with density $\rho_0(u)$. The meaning of the measure \mathbb{R} will be examined at a later point. On this probability space, consider the following information process

$$\xi_t = \int_0^t v_s(X)ds + \beta_t, \quad (15)$$

where $\{\beta_t\}$ is an \mathbb{R} -Brownian motion, independent of X . The task of the 'observer' is thus to determine the best estimate of X given the data $\{\xi_s\}_{0 \leq s \leq t}$. Assuming that the criteria for optimality is an estimator that minimises the quadratic error, standard results in filtering theory (cf. Wonham 1965, Liptser and Shiryaev 2001) show that the best estimate for X is the expectation of X with respect to the *a posteriori* density:

$$\frac{d}{du} \mathbb{R}(X < u | \mathcal{F}_t) = \frac{\rho_0(u) \exp\left(\int_0^t v_s(u)d\xi_s - \frac{1}{2} \int_0^t v_s^2(u)ds\right)}{\int_0^\infty \rho_0(u) \exp\left(\int_0^t v_s(u)d\xi_s - \frac{1}{2} \int_0^t v_s^2(u)ds\right) du}, \quad (16)$$

where $\mathcal{F}_t = \sigma(\{\xi_s\}_{0 \leq s \leq t})$. Notice that the right side of (16) is in fact identical to the right side of (11), provided that the measure change between \mathbb{P} and \mathbb{R} are suitably defined (recall that $\{\xi_t\}$ in the \mathbb{P} measure is a Brownian motion, while in the \mathbb{R} measure it is a drifted Brownian motion of (15)).

The above-specified filtering problem leads to the following probabilistic interpretation for the pricing kernel. Writing

$$N_t = \int_0^\infty \rho_0(u) \exp \left(\int_0^t v_s(u) d\xi_s - \frac{1}{2} \int_0^t v_s^2(u) ds \right) du \quad (17)$$

for the normalisation of the conditional density $\{\rho_t(u)\}$, we see that the pricing kernel is given by the ‘unnormalised’ conditional probability that $X > t$:

$$\pi_t = N_t \mathbb{R}_t(X > t), \quad (18)$$

where for simplicity we have written $\mathbb{R}_t(-) = \mathbb{R}(-|\mathcal{F}_t)$ for the conditional probability. Further, the price of a discount bond admits a probabilistic representation in the \mathbb{R} measure:

$$P_{tT} = \frac{\mathbb{R}_t(X > T)}{\mathbb{R}_t(X > t)}. \quad (19)$$

This formula shows that the price process of a discount bond is given by the ratio of the \mathbb{R} -conditional probability that the positive random variable X taking values greater than T and that of X taking values greater than t . By use of the Bayes formula, we deduce that (19) can alternatively be expressed in the form

$$P_{tT} = \mathbb{R}_t(X > T | X > t), \quad (20)$$

since the set $\{X > t\}$ contains the set $\{X > T\}$. This is essentially the representation obtained by Brody and Friedman (2009) for the discount bond using the information-based approach to interest rate modelling.

It is worth remarking that the random variable X has the dimension of time. In Brody and Friedman (2009), X was interpreted as the arrival time of liquidity crisis, in the narrow sense of a cash demand. Hence, under this interpretation, (20) shows that the bond price at t is the probability that the timing of the occurrence of a cash demand is beyond T , given that it has not yet occurred at t , and given the noisy information (15) concerning the value of X , in a suitably chosen measure \mathbb{R} .

4 Back to the market measure

The normalisation $\{N_t\}$ can be used to effect a measure change $\mathbb{R} \rightarrow \mathbb{P}$. To see this, note first that the process $\{W_t\}$ defined by

$$W_t = \xi_t - \int_0^t \mathbb{E}_s^{\mathbb{R}}[v_s(X)] ds \quad (21)$$

is an \mathbb{R} -Brownian motion with respect to the filtration $\{\mathcal{F}_t\}$ generated by the information process (15). In fact, this is just the innovations representation for the filtering problem posed above. Thus, the Brownian property can be verified

by checking that $\{W_t\}$ satisfies the martingale condition:

$$\begin{aligned}
\mathbb{E}_t^{\mathbb{R}}[W_T] &= \mathbb{E}_t^{\mathbb{R}} \left[\int_0^T v_s(X) ds + \beta_T - \int_0^T \mathbb{E}_s^{\mathbb{R}}[v_s(X)] ds \right] \\
&= \mathbb{E}_t^{\mathbb{R}} \left[\int_0^t v_s(X) ds + \beta_t - \int_0^t \mathbb{E}_s^{\mathbb{R}}[v_s(X)] ds \right] \\
&\quad + \mathbb{E}_t^{\mathbb{R}} \left[\int_t^T v_s(X) ds + (\beta_T - \beta_t) - \int_t^T \mathbb{E}_s^{\mathbb{R}}[v_s(X)] ds \right] \\
&= W_t,
\end{aligned} \tag{22}$$

where we have made use of the martingale property $\mathbb{E}_t^{\mathbb{R}}[\mathbb{E}_s^{\mathbb{R}}[v_s(X)]] = \mathbb{E}_t^{\mathbb{R}}[v_s(X)]$ of the conditional expectation for $s > t$, and the tower property of conditional expectation to deduce $\mathbb{E}_t^{\mathbb{R}}[\beta_T] = \mathbb{E}_t^{\mathbb{R}}[\beta_t]$. Along with $(dW_t)^2 = dt$, Lévy's characterisation shows that $\{W_t\}$ is an \mathbb{R} -Brownian motion.

On the other hand, an application of Ito's lemma on (17) gives

$$\frac{dN_t}{N_t} = \hat{v}_t d\xi_t, \tag{23}$$

from which it follows that

$$N_t = \exp \left(\int_0^t \hat{v}_s d\xi_s - \frac{1}{2} \int_0^t \hat{v}_s^2 ds \right) \tag{24}$$

is a positive martingale satisfying $N_0 = 1$. Hence $\{N_t\}$ can be used as the likelihood process to change the probability measure. Specifically, for any \mathcal{F}_t -measurable random variable Z_t we have

$$\mathbb{E}_s^{\mathbb{R}}[Z_t] = \frac{1}{N_s} \mathbb{E}_t^{\mathbb{P}}[N_t Z_t] \quad \text{and} \quad \mathbb{E}_s^{\mathbb{P}}[Z_t] = N_s \mathbb{E}_t^{\mathbb{R}} \left[\frac{1}{N_t} Z_t \right]. \tag{25}$$

In particular, (21) and (24) shows that $\{\xi_t\}$ is a Brownian motion under the \mathbb{P} measure. In addition, we find that the random variable X has the same probability law under \mathbb{P} as under \mathbb{R} , and that X and ξ_t are \mathbb{P} -independent. These properties can be verified by showing

$$\mathbb{E}^{\mathbb{P}}[e^{i(x\xi_t + yX)}] = \mathbb{E}^{\mathbb{P}}[e^{ix\xi_t}] \mathbb{E}^{\mathbb{P}}[e^{iyX}] \tag{26}$$

for all real x, y , and calculating the right side explicitly.

We remark that the conditional probability $\mathbb{R}_t(X > t)$ appearing in (18) can be interpreted as representing the pricing kernel in the \mathbb{R} measure. Specifically, writing Π_t for $\mathbb{R}_t(X > t)$, we deduce from (16) that

$$\Pi_t = \frac{\int_t^\infty \rho_0(u) \exp \left(\int_0^t v_s(u) d\xi_s - \frac{1}{2} \int_0^t v_s^2(u) ds \right) du}{\int_0^\infty \rho_0(u) \exp \left(\int_0^t v_s(u) d\xi_s - \frac{1}{2} \int_0^t v_s^2(u) ds \right) du}. \tag{27}$$

A short calculation making use of (21) shows that the \mathbb{R} -pricing kernel (27) can be expressed manifestly in the Flesaker-Hughston representation:

$$\Pi_t = \int_t^\infty \rho_0(x) G_t(x) dx, \tag{28}$$

where $\{G_t(x)\}$ is a one-parameter family of positive \mathbb{R} -martingales:

$$G_t(x) = \exp \left(\int_0^t \tilde{v}_s(x) dW_s - \frac{1}{2} \int_0^t \tilde{v}_s(x)^2 ds \right), \quad (29)$$

and where $\tilde{v}_t(x) = v_t(x) - \mathbb{E}_t^{\mathbb{R}}[v_t(X)]$. The dynamical equation satisfied by the \mathbb{R} -pricing kernel therefore reads

$$\frac{d\Pi_t}{\Pi_t} = -r_t dt - (\hat{v}_t + \lambda_t) dW_t, \quad (30)$$

where $\hat{v}_t = \mathbb{E}_t^{\mathbb{R}}[v_t(X)]$ and $\lambda_t = -\mathbb{E}_t^{\mathbb{R}}[v_t(X)|X > t]$.

5 Indeterminacy of the risk premium

Returning to the \mathbb{P} -measure, we recall that once a parametric model for the martingale volatility $\{v_t(x)\}$ is chosen, then prices of derivatives will in general depend on this model choice. Hence $\{v_t(x)\}$ can be calibrated from derivative prices. The initial term structure density $\rho_0(u)$, on the other hand, can be calibrated from the initial yield curve. By substituting these ingredients in (10) we thus obtain a market implied risk premium, subject to the model choice. Of course, any tractable model is unlikely to fit all derivative prices. One can nevertheless ask whether it is possible to fix $\{v_t(x)\}$ in a hypothetical situation where one has access to the totality of liquidly-traded derivative prices and an unlimited computational resource, i.e. whether it is possible in principle to fix $\{v_t(x)\}$ unambiguously. Perhaps not surprisingly, the answer is negative.

To see this, suppose that the volatility of the Flesaker-Hughston martingale family is decomposed in the form

$$v_t(u) = \phi_t(u) - \alpha_t, \quad (31)$$

where the vector process $\{\alpha_t\}$ is independent of X , and has no parametric dependence on u . The minus sign here is purely a matter of convention. Then writing

$$L_t = \exp \left(- \int_0^t \alpha_s d\xi_s - \frac{1}{2} \int_0^t \alpha_s^2 ds \right), \quad (32)$$

we find that the pricing kernel takes the form

$$\pi_t = L_t \int_t^\infty \rho_0(u) e^{\int_0^t \phi_s(u) d\xi_s - \frac{1}{2} \int_0^t \phi_s^2(u) ds + \int_0^t \phi_s(u) \alpha_s ds} du. \quad (33)$$

It follows that the price at time t of a contingent claim, with payout $H_T = h(S_T)$ at $T > t$, is given by

$$\begin{aligned} H_t &= \mathbb{E}_t \left[\frac{\pi_T}{\pi_t} H_T \right] \\ &= \mathbb{E}_t \left[\frac{L_T \int_T^\infty \rho_0(u) e^{\int_0^T \phi_s(u) d\xi_s - \frac{1}{2} \int_0^T \phi_s^2(u) ds + \int_0^T \phi_s(u) \alpha_s ds} du}{L_t \int_t^\infty \rho_0(u) e^{\int_0^t \phi_s(u) d\xi_s - \frac{1}{2} \int_0^t \phi_s^2(u) ds + \int_0^t \phi_s(u) \alpha_s ds} du} H_T \right] \\ &= \mathbb{E}_t^\alpha \left[\frac{\int_T^\infty \rho_0(u) e^{\int_0^T \phi_s(u) d\xi_s - \frac{1}{2} \int_0^T \phi_s^2(u) ds + \int_0^T \phi_s(u) \alpha_s ds} du}{\int_t^\infty \rho_0(u) e^{\int_0^t \phi_s(u) d\xi_s - \frac{1}{2} \int_0^t \phi_s^2(u) ds + \int_0^t \phi_s(u) \alpha_s ds} du} H_T \right], \quad (34) \end{aligned}$$

where we have used $\{L_t\}$ as a density martingale to change the measure. Evidently, under the new measure \mathbb{P}^α , the process $\{\xi_t^\alpha\}$ defined by

$$\xi_t^\alpha = \xi_t + \int_0^t \alpha_s ds \quad (35)$$

is a standard Brownian motion. Substituting (35) in (34) we deduce that

$$H_t = \mathbb{E}_t^\alpha \left[\frac{\int_T^\infty \rho_0(u) e^{\int_0^T \phi_s(u) d\xi_s^\alpha - \frac{1}{2} \int_0^T \phi_s^2(u) ds} du}{\int_t^\infty \rho_0(u) e^{\int_0^t \phi_s(u) d\xi_s^\alpha - \frac{1}{2} \int_0^t \phi_s^2(u) ds} du} H_T \right], \quad (36)$$

which is identical to the pricing formula under the \mathbb{P} measure had $\{\alpha_t\}$ been identically zero in the first place, on account of the following observation. The price of the underlying asset at time T can be expressed in the form

$$S_T = S_0 \exp \left(\int_0^T \left(r_s + \lambda_s \sigma_s - \frac{1}{2} \sigma_s^2 \right) ds + \int_0^T \sigma_s d\xi_s \right), \quad (37)$$

where $\{\sigma_t\}$ is the volatility of $\{S_t\}$. Now if the volatility of the martingale family $\{M_t(u)\}$ takes the form (31), then the risk premium can be expressed as

$$\lambda_t = \lambda_t^\alpha + \alpha_t, \quad (38)$$

where $\{\lambda_t^\alpha\}$ is the risk premium in the \mathbb{P}^α measure:

$$\lambda_t^\alpha = - \frac{\int_t^\infty \rho_0(u) \phi_t(u) e^{\int_0^t \phi_s(u) d\xi_s^\alpha - \frac{1}{2} \int_0^t \phi_s^2(u) ds} du}{\int_t^\infty \rho_0(u) e^{\int_0^t \phi_s(u) d\xi_s^\alpha - \frac{1}{2} \int_0^t \phi_s^2(u) ds} du}. \quad (39)$$

Substituting (35) and (38) in (37), we obtain

$$S_T = S_0 \exp \left(\int_0^T \left(r_s + \lambda_s^\alpha \sigma_s - \frac{1}{2} \sigma_s^2 \right) ds + \int_0^T \sigma_s d\xi_s^\alpha \right). \quad (40)$$

We thus find that the probability law of the random variable S_T , and hence H_T , under the \mathbb{P} -measure with $\alpha_t = 0$, is the same as that under the \mathbb{P}^α -measure with $\alpha_t \neq 0$. It follows that any *addition* of terms in the martingale volatility $\{v_t(u)\}$ that is independent of the parameter u does not affect current price levels.

The above result shows that the risk premium vector $\{\lambda_t\}$ can be determined from market prices of derivatives only up to an additive vectorial term $\{\alpha_t\}$. This freedom, however, is not arbitrary; it can only arise from a constant (i.e. independent of the parameter u) addition to the volatility of the martingale family in the form of (31).

6 Information-based interpretation

The ambiguity in the determination of the risk premium can be interpreted from the viewpoint of information-based asset pricing theory of Brody *et al.* (2007). In the information-based pricing framework one models the market filtration

directly in the form of an information process concerning market factors relevant to the cash flows of a given asset. Our objective here, which extends the previous work of Brody and Friedman (2009), is to analyse the model (15) for the information process that determines the pricing kernel.

The interpretation of the information process (15) is as follows. Market participants are concerned with the realised value of the random variable X , which, in a certain sense can be interpreted as the timing of a serious liquidity crisis. In reality, market participants observe price processes, or equivalently the underlying Brownian motion family $\{\xi_t\}$. As indicated above, under the physical \mathbb{P} -measure the random variables X and ξ_t are independent. However, market participants ‘perceive’ information with certain risk adjustments characterised by the density martingale $\{N_t\}$ of (24). In this risk-adjusted measure, the path $\{\xi_t\}$ represents the aggregate of noisy information for the value of X in the form of (15). The ‘signal’ concerning the value of X , in particular, is revealed to the market through the structure function $\{v_t(u)\}$, which in turn determines the volatility structure of the pricing kernel, and hence the risk premium.

Suppose that the structure function $\{v_t(u)\}$ takes the form (31), where $\{\alpha_t\}$ is independent of X . Then because (15) represents the information process for the random variable X , the constant $\{\alpha_t\}$ combines with the ‘noise’ term $\{\beta_t\}$. In other words, the choice of $\{\alpha_t\}$ is entirely equivalent to the choice of noise; the Brownian noise is replaced by a drifted Brownian noise. This change of noise composition does not affect current asset prices, and therefore is not directly detectable from market data, even though asset-price drifts are modified, in general in an unidentifiable manner. Note that the point of view that the indeterminacy of the asset price drifts is caused by noise has been put forward heuristically by Black (1986); our observation thus formalises this argument more precisely.

It is worth remarking briefly the observation made in Brody and Friedman (2009) concerning the form of the structure function $\{v_t(u)\}$ in the absence of the noise drift $\{\alpha_t\}$. Since small values of X imply imminent liquidity crisis, in an ideal market the signal-to-noise ratio of the information process (15) should be large for small values of X , as compared to large values of X . In other words, under normal market conditions we expect the signal magnitude $|v_t(u)|$ be decreasing in u for every t . Conversely, if $|v_t(u)|$ is increasing in u , then the excess rate of return above the short rate for discount bonds, i.e. the inner product of the risk premium and the discount bond volatility, is negative, yielding negative excess rate of return due to the inverted form of the structure function $\{v_t(u)\}$.

7 Anomalous price behaviour

The fact that current asset prices are unaffected by changes in the structure of the noise term does not imply that $\{\alpha_t\}$ can be ignored altogether. Indeed, (38) shows that the existence of such a component does shift the risk premium. Since the drift of an asset with volatility $\{\sigma_t\}$ is given in the \mathbb{P} -measure by $r_t + \lambda_t \sigma_t$, the noise-induced drift $\alpha_t \sigma_t$ can generate various anomalous price dynamics under the physical \mathbb{P} -measure.

As an example, let us consider the case of an anomalous price growth, or a bubble. In the large vector space of asset volatilities, it is inevitable that

volatility vectors form clusters consisting of different sectors or industries. This is because, by definition, a given sector of companies share analogous risk exposures. Now if an anomalous noise component $\{\alpha_t\}$ at some point in time emerges to point in the direction of one of these volatility clusters, then this can cause a sharp rise in the share prices of that sector. Since the noise vector $\{\alpha_t\}$ carries no real economic information, this can be identified as a bubble, where prices of a set of assets grow sharply, and independently of the ‘true’ state of affairs, without seriously affecting price processes of other assets. Similarly, at a later time, the magnitude of $\{\alpha_t\}$ can diminish. In particular, more reliable information concerning the true state of affairs may be revealed, which in turn leads to an increase in the magnitudes of volatilities on the one hand, while on the other hand the risk premium vector can point in a direction such that the inner product $\lambda_t \sigma_t$ takes a large negative value; thus leading to a bubble ‘burst’.

In the finance and economics literature, there exists a substantial work on the study of various aspects of financial bubbles (see, e.g., Camerer 1989 for an early review). It is important to note that our characterisation of a bubble is motivated by an information-based perspective. One commonly used definition of a bubble, on the other hand, is given by the difference between the current price and the expected discounted future cash flows in the risk-neutral measure (cf. Tirole 1985, Heston *et al.* 2007). Under this definition, discounted asset prices in the risk-neutral measure can be modelled by use of strict local martingales (Cox and Hobson 2005, Jarrow *et al.* 2007, 2010), within the arbitrage-free pricing framework.

While this formulation of a bubble leads to the unravelling of many interesting mathematical subtleties underlying fundamental theorems of asset pricing, from an information-theoretic viewpoint the plausibility of such a definition for a bubble seems questionable. In particular, a mathematical definition of a financial bubble that involves no reference to the \mathbb{P} measure seems restrictive; a bubble, after all, is a phenomenon seen under the \mathbb{P} measure. The pricing kernel approach, on the other hand, is based on a stronger assumption that if $\{S_t\}$ represents the price process of a liquidly traded asset, then $\{\pi_t S_t\}$ must be a true \mathbb{P} -martingale. As such, the discounted \mathbb{Q} -expectation of future asset price necessarily agrees with the current value, or else there are arbitrage opportunities.

The conventional definition of a financial bubble in terms of the inequality $S_t > \pi_t^{-1} \mathbb{E}_t[\pi_T S_T]$ is sometimes justified heuristically by the fact that some traders, when they are under the impression that there is a bubble and thus traded prices are above the ‘fundamental’ values, will nevertheless participate in the apparent bubble with the view that they can withdraw from their positions before the crunch (see, e.g., Camerer 1989 and references cited therein). This example and other similar ones are often used in support of the argument that some traders are willing to purchase stocks even when they know that the price level is above its fundamental value. The shortcomings in such an argument are that (a) the role of market filtration is not adequately taken into account; and that (b) the fact that such a stock purchase is equivalent to the purchase of an American option is overlooked. A more plausible characterisation of a bubble participation seems to be as follows. Given the information $\{\mathcal{F}_t\}$, a trader estimates that there is a bubble that will continue to grow for a while. Hence, subject to the filtration, the best estimate of the future cash flow for this trader, with a suitable risk-adjustment, is given by $\sup_{\tau} \pi_t^{-1} \mathbb{E}_t[\pi_{\tau} S_{\tau}]$, where τ is

a stopping time when the stock is sold. If this expectation agrees to the current price level, then a transaction occurs. Conversely, it seems implausible that a transaction takes place if the best estimate by a rational trader of a discounted cash flow is lower than the current price level.

The view we put forward here is that a bubble in an asset ought to be identified with an anomaly in the rate of return of that asset, and not with an anomaly in the price level itself. Here, a precise definition of an ‘anomaly’ in the drift is essentially what we have described above, namely, the existence of an additive term in the volatility of the martingale family $\{M_t(u)\}$ that is constant in the parameter u . Based on this definition, it is admissible that price processes behave in a manner that does not always reflect what one might perceive as the true state of affairs, had one possessed better information concerning the true worth of the assets. Put the matter differently, decisions concerning transactions that ultimately lead to price dynamics are made in accordance with the unfolding of information. Since this information is necessarily noisy, the best filters chosen by market participants will inevitably deviate from true values of assets being priced. If the noise structure changes, then it is only reasonable that the dynamical aspects of these deviations will likewise change. In particular, the increment of the innovations representation—that characterises the arrival of ‘real’ information over the interval $[t, t + dt]$ —is given by

$$dW_t = d\xi_t - \hat{\phi}_t dt + \alpha_t dt, \quad (41)$$

where $\hat{\phi}_t = \mathbb{E}_t^{\mathbb{R}}[\phi_t(X)]$, and this illustrates in which way the existence of a nonzero noise drift $\{\alpha_t\}$ affects the dynamics.

Our characterisation of anomalous price dynamics is not confined to the consideration of financial bubbles. Again, in the large vector space of asset volatilities, it seems plausible that equity market volatilities and fixed-income volatilities generally lie on distinct subspaces. If the noise vector $\{\alpha_t\}$ has a tendency to lie in the direction of equity-volatility subspace, then this naturally leads to an excess growth in the equity market, explaining the phenomena of the so-called equity premium puzzle, where over time the rate of return associated with the equity market considerably exceeds that of the bond market (see, e.g., Kocherlakota 1996 for a review).

8 Relation to the risk-neutral measure

We have established the relation between the auxiliary probability measure \mathbb{R} and the physical measure \mathbb{P} . The relation between the latter and the risk-neutral measure \mathbb{Q} , on the other, involves the risk premium process $\{\lambda_t\}$. To recapitulate these two relations, we have

$$dW_t = d\xi_t - \hat{v}_t dt \quad \text{and} \quad dW_t^* = d\xi_t + \lambda_t dt, \quad (42)$$

where

$$\hat{v}_t = \frac{\int_0^\infty \rho_0(u) v_t(u) M_t(u) du}{\int_0^\infty \rho_0(u) M_t(u) du} \quad \text{and} \quad \lambda_t = -\frac{\int_t^\infty \rho_0(u) v_t(u) M_t(u) du}{\int_t^\infty \rho_0(u) M_t(u) du}, \quad (43)$$

and where we let $\{W_t^*\}$ denote the \mathbb{Q} -Brownian motion. By combining the two relations in (42) we deduce at once that the measure-change density martingale

is given by

$$\left. \frac{d\mathbb{Q}}{d\mathbb{R}} \right|_{\mathcal{F}_t} = \exp \left(- \int_0^t (\hat{v}_s + \lambda_s) dW_s - \frac{1}{2} \int_0^t (\hat{v}_s + \lambda_s)^2 ds \right), \quad (44)$$

which determines the general relation between \mathbb{Q} and \mathbb{R} .

As indicated above, a closer inspection on (43), however, shows that

$$\hat{v}_t = \mathbb{E}_t^{\mathbb{R}}[v_t(X)] \quad \text{and} \quad \lambda_t = -\mathbb{E}_t^{\mathbb{R}}[v_t(X)|X > t]. \quad (45)$$

In other words, under the restriction $X > t$, we have, conditionally, $\hat{v}_t + \lambda_t = 0$. Therefore, the auxiliary measure \mathbb{R} , whose existence is ensured by the lack of arbitrage and the existence of pricing kernel, can be interpreted as an extension of the risk-neutral measure. Conversely, by restricting to the event $X > t$, we can think of \mathbb{R} indeed as the risk-adjusted measure.

9 Stochastic volatility

So far we have analysed the case for which $\{v_t(x)\}$ is a deterministic function of time. The fact that the volatility structure of the martingale family $\{M_t(x)\}$ is deterministic, however, does not imply deterministic volatilities for asset prices. On the contrary, even for an elementary discount bond, the associated volatility process is highly stochastic. Hence when we speak about a ‘stochastic volatility’ we have in mind the volatility for the martingale family $\{M_t(x)\}$, whereas the stochasticity for asset prices is presumed.

From the viewpoint of practical implementation, it probably suffices to restrict attention to deterministic volatility structures, since deterministic volatilities for $\{M_t(x)\}$ give rise to a range of sophisticated stochastic volatility models for asset prices. Indeed, it is shown in Brody *et al.* (2011) that even in the very restricted case of a single factor model with the time-independent volatility $v_t(x) = e^{-\sigma x}$ that depends only on one model parameter σ , it is possible to calibrate caplet prices across different maturities reasonably accurately.

It is nevertheless of interest to enquire whether the auxiliary information process exists in the more general context of stochastic volatilities. For this purpose, let us begin by considering the case where $\{v_t(x)\}$ admits the decomposition (31) and where $\{\phi_t(x)\}$ is deterministic and $\{\alpha_t\}$ is chosen such that the noise term $n_t \equiv \int_0^t \alpha_s ds + \beta_t$ is an $\{\mathcal{F}_t^\beta\}$ -measurable Gaussian process. Then an application of the martingale representation theorem shows that $\{n_t\}$ admits a decomposition of the form

$$n_t = \int_0^t b_s ds + \int_0^t \gamma_s d\beta_s, \quad (46)$$

where $\{b_s\}$ and $\{\gamma_s\}$ are deterministic. A short calculation then shows that an auxiliary information process

$$\xi_t = \int_0^t \phi_s(X) ds + n_t \quad (47)$$

in the \mathbb{R} measure indeed exists, with the property that the scaled information process $\int_0^t \gamma_s^{-1} d\xi_s$ determines the market Brownian motion and that $\{b_t\}$ plays

the role similar to that of a deterministic $\{\alpha_t\}$ in the previous analysis, and hence is not determinable from current market prices.

The foregoing example shows how one can model the random rise and fall of anomalous price dynamics. More generally, the structure function $\{v_t(x)\}$ can depend in a general way on the history of the information process up to time t . In this case, we obtain a generic stochastic volatility model for the martingale family. Provided that the structure function is sufficiently well behaved so that relevant stochastic integrals exist, the auxiliary information process can be seen to exist in the \mathbb{R} measure. To illustrate this, consider an elementary ‘toy model’ for which information process takes the form of an Ornstein-Uhlenbeck process:

$$\xi_t = e^{\sigma X t} \int_0^t e^{-\sigma X s} d\beta_s, \quad (48)$$

where σ is a parameter, and X and $\{\beta_t\}$ are independent. Such an information process corresponds to a stochastic volatility model for which the volatility process is given by a linear function of the \mathbb{P} -Brownian motion: $v_t(x) = \sigma x \xi_t$.

10 Discussion

The main results of the paper are as follows: We have derived the existence of an auxiliary filtering problem underlying arbitrage-free modelling of the pricing kernel; the solution of which determines the volatility structure of the positive martingale family $\{M_t(u)\}$ appearing in the Flesaker-Hughston representation for the pricing kernel. We have demonstrated that the structure of the ambient information process fully characterises the risk premium process $\{\lambda_t\}$. We have shown, under the Brownian-filtration setup, that $\{\lambda_t\}$ admits a canonical decomposition into two terms in an additive manner; the systematic term that can be calibrated from current market data for derivative prices, and the idiosyncratic term that cannot be estimated (unless, of course, one can estimate drift processes of risky assets), and thus can be identified as noise.

It is worth emphasising that these results hold irrespective of our choice of interpretation. Nevertheless, our characterisation of anomalous price dynamics seems sufficiently compelling, for, such phenomena are ultimately observed under the physical measure \mathbb{P} . One might ask what causes the evolution of the noise drift $\{\alpha_t\}$. This is an interesting econometric question that, however, goes beyond the scope of the present investigation. It suffices to remark that the random variable X that constitutes the signal component of the ambient information process has units of time, and thus is ultimately linked to the term structure of financial markets. One possible explanation of the excess equity premium therefore is that fixed-income market intrinsically embodies more information concerning the term structure as compared to the equity market, and this imbalance is manifested in the form of an additional drift in the noise component pointing generally towards the direction of equity volatility vectors.

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References

- Björk, T. (2009). *Arbitrage Theory in Continuous Time*. Oxford University Press, third edition.
- Black, F. (1986). Noise. *Journal of Finance* **41** 529–543.
- Brody, D. C. and Hughston, L. P. (2001). Interest rates and information geometry. *Proceedings of the Royal Society London* **457** 1343–1363.
- Brody, D. C., Hughston, L. P. and Macrina, A. (2007). Beyond hazard rates: a new framework for credit-risk modelling. In Elliott, R., Fu, M., Jarrow, R., and Yen, J. Y., editors, *Advances in Mathematical Finance, Festschrift volume in honour of Dilip Madan*. Birkhäuser and Springer.
- Brody, D. C. and Friedman, R. F. (2009). Information of interest. *Risk* **December** 101–106.
- Brody, D. C., Hughston, L. P. and Mackie, E. (2011). Calibration of geometric Lévy interest rate models. Imperial College Working Paper.
- Camerer, C. (1989). Bubbles and fads in asset prices. *Journal of Economic Surveys* **3** 3–41.
- Cox, A. M. and Hobson, D. G. (2005). Local martingales, bubbles and option prices. *Finance and Statistics* **9** 477–492.
- Flesaker, B. and Hughston, L. P. (1996). Positive interest. *Risk* **November** 46–49.
- Flesaker, B. and Hughston, L. P. (1997). International models for interest rates and foreign exchange. *Net Exposure* **3** 55–79.
- Kocherlakota, N. R. (1996). The equity premium: It’s still a puzzle. *Journal of Economic Literature* **34** 42–71.
- Jarrow, R. A., Protter, P. and Shimbo, K. (2007). Asset price bubbles in complete markets. In Fu, M., Jarrow, R., Yen, J.-Y., and Elliott, R., editors, *Advances in Mathematical Finance, Applied and Numerical Harmonic Analysis*, pages 97–121. Birkhäuser Boston.
- Jarrow, R. A., Protter, P. and Shimbo, K. (2010). Asset price bubbles in incomplete markets. *Mathematical Finance* **20** 145–185.
- Liptser, R. S. and Shiryaev, A. N. (2001). *Statistics of Random Processes*. Berlin: Springer-Verlag, second edition.
- Rogers, L. C. G. (1997). The potential approach to the term structure of interest rates and foreign exchange rates. *Mathematical Finance* **7** 157–176.
- Rogers, L. C. G. (2001). The relaxed investor and parameter uncertainty. *Finance and Stochastics* **5** 131–154.
- Tirole, J. (1985). Asset bubbles and overlapping generations. *Econometrica* **53** 1499–1528.
- Wonham, W. M. (1965) Some applications of stochastic differential equations to optimal nonlinear filtering. *J. SIAM A* **2** 347–369.